Teaching Linear Algebra in the 21st Century

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High School Capstone

May 24, 2021

I dedicate this paper to my loving parents, Debbie and Eric.

Abstract

Linear algebra is of vital importance to the modern world, but how to best teach it remains a subject of hot debate. To focus on theory or applications? Emphasize animation and interactivity? Shun determinants? Teach proofs? Incorporate mastery-based learning? We explore these questions in order to design a linear algebra class fit for the 21st century. We then teach this class online in the Spring of 2021, finding great success with our new techniques. We recommend adopting these more broadly.

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Chapter 1: Introduction and Literature Review

Linear algebra has stunning applications in the modern world, from artificial intelligence and internet search to quantum physics and weather prediction, not to mention forming the backbone of many other fields of pure math. Renowned professor Gilbert Strang calls linear algebra "*the* most important subject in college mathematics."[1] The world needs experts who understand linear algebra, which places great importance on how we teach it. Despite that, professors and students alike bemoan the "frustrating experience" of teaching linear algebra.[2] Since the 1940s, students have struggled with formalism and drowned in matrix calculations.

These two difficulties reflect the two versions of linear algebra commonly taught today: a computational course, focusing on matrices and Euclidean space, and a theoretical course, focusing on vector spaces and proofs. Both are surprisingly recent in history. While the roots of linear algebra lie in the millenia-old problem of solving systems of equations, its modern, axiomatic form arose only around 1930.[3] Linear algebra was widely adopted in the undergraduate curriculum only in the 1960s,[4] around the time computers rendered it relevant. Students in U.S. universities often take a matrix class first, then a second course that focuses on theory. Each type of linear algebra class has associated challenges to teaching.

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[1] Gilbert Strang. *Linear Algebra: A Happy Chance to Apply Mathematics*. Department of Mathematics, MIT, Cambridge, MA 02139, USA. Recent Advances in Computational Sciences. July 2008: 346-356.
[2] Dorier. *On the Teaching of Linear Algebra*. Kluwer Academic Publishers, 2002: 191.
[3] Dorier: 30.
[4] Alan Tucker. "The Growing Importance of Linear Algebra in Undergraduate Mathematics." Vol. 24, No. 1 (Jan., 1993): pp. 7.

One challenge for the matrix-based course is to balance applications and core content. With uncountable ways to use linear algebra in the real world, and nearly uncountable topics to cover in the theory, professors face a difficult choice of what to omit.[5] Another challenge of a first course is a common feeling among students that the material does not connect to what they learned in high school.[6] Perhaps the most significant challenge is that matrix arithmetic is tedious, error-prone, and not particularly insightful into the geometry behind the scenes. For that reason, scholars such as Ghislaine Gueudet have researched the benefits and pitfalls of using geometry to teach linear algebra.[7] Many courses now include computer visualizations and programming to encourage a geometric understanding and also to allow students to experiment with applications.[8]

The main challenge for the theory-based class is that students are generally ill-prepared in set theory and formal proofs. France for a time experimented with teaching such formalism in high school, but soon reverted.[9] Students in France, the United States, and beyond thus enter college with scant instruction, if any, in these prerequisites. With such emphasis placed throughout high school on heads-down calculation[10] and multiple-choice standardized tests, the transition to proof-writing may be especially a shock. Students face a triple challenge, in summary: to learn linear algebra, set theory,

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[5] Sinan Aydin. “The factors effecting (sic) teaching linear algebra.” Elsevier, Social and Behavioral Sciences, Volume 1, Issue 1, 2009: 1549-1553.
[6] Dorier: 179.
[7] Ghislaine Gueudet. “Using Geometry to Teach and Learn Linear Algebra.” Research in Collegiate Mathematics Education, 2006, VI, pp.171-195.
[8] Carl C. Cowen. “The Centrality of Linear Algebra.” Mathematical Association of America. Accessed March 2021. www.maa.org/centrality-of-linear-algebra.
[9] Dorier: 85.
[10] Katherine K Merseth. “How Old is the Shepherd? An Essay About Mathematics Education” (1993).

and proof writing, all at once. An additional challenge is that the modern slickness of presentation can often hide depth. While the axioms rose out of centuries of debate and slow unification, students learn them in weeks and often without context. Professors who regard material as elementary are sometimes baffled at student misunderstandings, an issue some propose could be remedied by instructing teachers in the history of the subject, so as to give some "'meat' to the 'bare bones' of axiomatic approach."[11]

Pedagogical approaches such as mastery-based learning[12] could also help students to learn linear algebra, although to our knowledge no one has conducted a specific experiment to test it. Some have experimented with ideas such as a flipped classroom for teaching linear algebra.[13] Others have implemented spaced repetition in different ways.[14] Regardless of how linear algebra is taught, there is the question of when and how much. Most math programs in American universities devote the first year entirely to calculus, leading many to call for a greater emphasis on linear algebra early on,[15] one which would place the subject on equal footing with calculus.[16]

In 1995, American mathematician Sheldon Axler proclaimed that linear algebra courses should ban determinants, which he described as "difficult, non-intuitive, and

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[11] Dorier: 59.
[12] Salman Khan. *The One World Schoolhouse*. New York: Twelve, 2012: 37-43.
[13] Robert Talbert. "Inverting the Linear Algebra Classroom." PRIMUS, Volume 24, 2014, Issue 5: 361-374.
[14] Ben Tilly. "Teaching Linear Algebra." Accessed March 2021. bentilly.blogspot.com/2009/09/teaching-linear-algebra.html; Matthew Simonson. "Eigenightmares, Dancing Stick Figures, and the Advantages of a Spiral Approach to Pedagogy." American Mathematical Society Blogs. blogs.ams.org/mathgradblog/2016/04/06/eigenightmares-dancing-stick-figures-advantages-spiral-approach-pedagogy/
[15] Drew Armstrong. "More Linear Algebra, Please." American Mathematical Society Blogs. Accessed March 2021. blogs.ams.org/matheducation/2016/09/19/more-linear-algebra-please
[16] Tucker: 6.

often defined without motivation."[17] He soon spread his doctrine by publishing the now-popular textbook *Linear Algebra Done Right*, which is aimed to guide a theory-based course. Axler claims his determinant-free proofs are clearer and more insightful.

With the rise of computers, some researchers have begun to explore new interfaces for math beyond pencil and paper. Symbolic reasoning is a human superpower, but beyond symbols are the ideas students actually need to learn. In his provocatively titled essay "Kill Math," computer scientist Bret Victor describes his goal as a "widely-usable, insight-generating alternative to symbolic math."[18] Researcher Michael Nielson went so far as to develop a prototype medium for not only explaining but exploring linear algebra.[19] These ideas remain mostly untouched in the classroom.

Perhaps the most well-known recent innovation in explaining linear algebra is animation. Video can bring to life the rich visual side of linear algebra concepts. Most famous is the YouTube series *Essence of Linear Algebra* by Grant Sanderson, which has amassed more than eleven million views.[20] Sanderson's animations frequently visualize the effect of a linear transformation on the plane. The series is widely celebrated by the mathematical community.

In this paper, we combine history with recent innovations to design a linear algebra course fit for the 21st century. We focus on theory and deemphasize matrix calculations in order to not lose students in a sea of calculation. We introduce

[17] Sheldon Axler. "Down with Determinants!" The American Mathematical Monthly, 102. February 1995.
[18] Bret Victor. "Kill Math." 11 April 2011. worrydream.com/KillMath
[19] Michael Nielson. "Toward an Exploratory Medium for Mathematics." February 2016. cognitivemedium.com/emm/emm.html
[20] 3blue1brown. *Essence of Linear Algebra*. YouTube. Accessed 4 March 2021.

fundamental concepts in the first two weeks via Grant Sanderson's animations to build visual mental models in students and to lend the feeling of solid ground in an otherwise unfamiliar and abstract environment. We offer explorable playgrounds to students. We also encourage students to explore applications that interest them. We incorporate mastery-based learning and innovative video feedback. In the Spring of 2021, we delivered the course to six students online, during the global COVID-19 pandemic. At the end, we reflect on the experience.

Chapter 2: Methods

The experimental linear algebra course ran for 19 weeks. We co-taught the class with Chris Lippi, the Khan Lab School statistics teacher. Originally seven students signed up, but one dropped after a week due to a busy schedule. The rest signed a contract confirming they understood the course was experimental and taught by a high school student.

We used the third edition of the textbook *Linear Algebra Done Right* by Sheldon Axler. Homework was copied and modified from the publicly available linear algebra class 18.700 via MIT OpenCourseWare. We favored abstract theory over matrix calculations, but in line with 18.700 we included a brief supplement on matrix algorithms which Axler omitted. From the beginning, we clarified to the six students that any and all feedback was valuable. In many instances, we adopted student ideas. Each week followed a similar formula with class on Monday, Wednesday, and Friday. See the syllabus included in Appendix A for the full schedule.

Mondays were typically spent in an interactive lecture covering the concepts from the week's reading. One issue in many classrooms is that a small number of speakers dominate whenever the teacher asks a question, while the majority of students will stay silent. To solve this problem, we called on students at random for the majority of the questions. One student helpfully provided a short Python program to automate the process. We modified it to more evenly distribute names and now use it in most Monday classes. See Appendix B for the code. The typical format of the lesson was to go through three to five pages of handwritten notes, interspersed with questions and examples. We used the app PDF Expert to write the notes and to continue annotating live during the class, using the blank spaces on the notes as a virtual whiteboard. We ended Monday classes with a brief “What’s Coming Up” section to keep students on track remembering assignments.

Wednesdays were typically supported group work time for the students. The built-in work time not only alleviated the pressure of the high work load, but also allowed students to work together on problems. About every other week, we met one-on-one with each student for five to ten minutes. The one-on-one sessions helped to keep track of which students were struggling and to field feedback on how to improve the class. We recorded student responses in a spreadsheet for later reference.

Fridays were typically office hours. After low attendance the first week, however, we rebranded Fridays as Hallway Time in memory of the lost minutes spent chatting in hallways between classes, before the COVID-19 pandemic forced virtual school. Additionally, we invented an original joke each week which students could only hear if they came to Hallway Time. See Appendix C for the jokes delivered each week. Attendance soon jumped to a little over half the class on an average Friday. Most students used Hallway Time as a chance to work silently, with the teacher available if questions arose.

The first two weeks followed a different format, because we felt a gentler on-ramp could help prepare students for the abstract material they would encounter in the textbook. Students watched the *Essence of Linear Algebra* series by Grant Sanderson on YouTube across the first two weeks. To further solidify visual intuition, we spent Monday classes discussing questions related to the ideas in the videos. We used the random caller code to vary who answered questions. Before class, students answered these questions on their own. We decided against multiple choice because it not only requires more work on the teacher's part to create wrong answers, but it requires less reflection on the student's part to answer. Instead students wrote their answers free form. The survey ended with a "PANIC BUTTON" so that we could intervene if any students felt lost. We also meant for the panic button to suggest a supportive environment to students. The first Wednesday class these first two weeks was dedicated to introductory set theory; the other was spent on proof techniques.

Homework was organized as follows after the first two weeks. Students typically read two sections of *Linear Algebra Done Right* by Wednesday. Students filled out a survey afterward, answering three to five questions that we composed in order for them to reflect on the concepts and gain a first working familiarity with them. Students received detailed feedback on their survey responses in which we corrected misconceptions. Each week's survey also contained a final question that was some variant of "Check-in: How are you feeling with the material?" Reading student responses gave us repeated chances to intervene if a student felt behind.

Problem sets were typically due every other week. We used the 9 problem sets publicly available from MIT OpenCourseWare 18.700, with two major changes. First, we replaced most problems relating to vector spaces over finite fields. We didn't have enough time in class to cover finite fields to a satisfactory depth. Second, from the beginning we made the final two problem sets optional. The reason was again a lack of time, given that we slowed down the pace compared to MIT's original speed of one problem set per week. We felt that the tradeoff of less material for more depth was desirable. Students could revise a problem set as many times as they wanted, until the next problem set was due.

Projects were typically due every month and a half. Students completed three projects over the 19 weeks. The purpose of the projects was to allow students to explore applications of the theory in areas that interested them. As such, we allowed students to choose project topics independently and bring them to us for approval. Students also gave a short presentation about their results in Wednesday class the week each project was due, with the goal of exciting their peers. Students were allowed a revision, but only one ever needed one. This format succeeded for some students, but others had poor skills in math communication. The free-form nature of the projects did not teach communication either. In an effort to directly remedy this, we redefined the final project to be a seminar, with the express purpose of practicing mathematical communication. Students read a two-part blog post about the singular value decomposition,[21] developed questions, and then discussed them.

Exams were spaced about one per month, with the twofold purpose of encouraging students to review old material and of allowing students more flexibility in how to earn mastery in the course. To reduce anxiety and promote a close-knit feel for

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[21] Jeremy Kun. "Singular Value Decomposition Part 1: Perspectives on Linear Algebra." Math ∩ Programming. Published April 2016; "Singular Value Decomposition Part 2: Theorem, Proof, Algorithm." Published May 2016. Each accessed May 2021.

jeremykun.com/2016/04/18/singular-value-decomposition-part-1-perspectives-on-linear-algebra
jeremykun.com/2016/05/16/singular-value-decomposition-part-2-theorem-proof-algorithm/

the class, we branded each exam differently in a humorous way. The first exam was called the I.M.P.S., for Individual Mini Problem Set. The second exam was called the S.M.I.L.E., for Synchronous Mini Individual List of Exercises. The third exam was called the S.E.E.Y.A.L.A.T.E.R., for Salutatory Exploratory Exercises You Answer Laughingly And Through Excellent Responses. Students were allowed to revise their exam once if desired. To revise, students completed one to two additional problems that required concepts from the entire unit.

Grading worked under a mastery-based system. Reading assignments were complete/incomplete, while problems sets, projects, and exams were graded as novice, emergent, or proficient. There were three units. To receive mastery in a unit, students needed a grade of proficient in all of the unit's problem sets and in the unit's project and exam. To receive mastery in the course, students needed to master each of the three units: Vector Spaces and Linear Maps, Matrices and Eigenvectors, and Inner Product Spaces and the Spectral Theorem. We also read material in Axler beyond the third unit, but it was ungraded and did not have associated mandatory homework.

We pioneered a new method for grading problem sets. The issue with typed feedback was that student submissions required more nuanced revisions than was easy to convey efficiently in writing. The ideal would have been to meet with each student individually to discuss their work. Neither was that feasible. The solution we chose was video feedback. Each student received a personalized video about ten minutes long detailing observations on their solutions to each problem that was meant to guide them on their revision. The videos took about thirty minutes each to produce, which is much less than it would take to type feedback. Additionally, students could pause and rewind the video, which would be impossible if we met with students synchronously. We used the recording app Loom because of its unique capability to instantly upload the video after recording. The video grading approach was largely enabled by the online environment of virtual schooling; downloading the PDFs that students submitted and marking them up with PDF Expert sped up the recording process compared to if students handed in their solutions on paper.

To guide students to begin on the problem sets, we created Get Started videos for each problem. The videos were little more than reading the problem, discussing the definitions, and often drawing a picture. The goal was to avoid students getting stuck even understanding the problems, which was a potential issue given the more abstract style of math we covered compared to the students' previous experiences.

To help keep students on top of assignments, we sent a Week Preview email every Sunday. The preview outlined assignments due in the coming two weeks. Each also repeated a common theme throughout the entire course: we want you to succeed, so reach out if you have questions, concerns, or ideas. We wrote the preview in an excited tone to remind students that linear algebra is fun, alive, and worth their time to learn.

We recorded the lectures and posted them unlisted on YouTube. The recordings are available at youtube.com/playlist?list=PLapqQU8bF_-_hXoCddMwi8ahlWLg9WQ_U.

Chapter 3: Results

In Appendix A, we gather together all material created for the class: syllabus, lecture notes, weekly surveys, problem sets, exams. These are the tangible results. We also wanted to know whether our new ideas had merit. To measure this, we asked all students to fill out a course evaluation.

The first question was to rate various course components, where 1 = hated and 5 = loved.

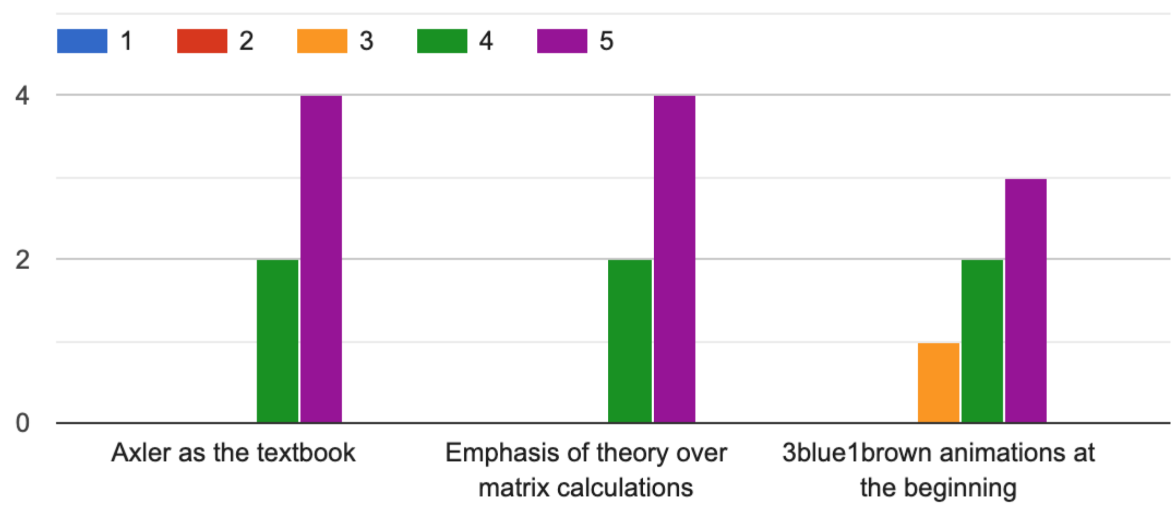

Figure 1a: blue = 1, red = 2, orange = 3, green = 4, purple = 5. Hated is 1. Loved is 5.

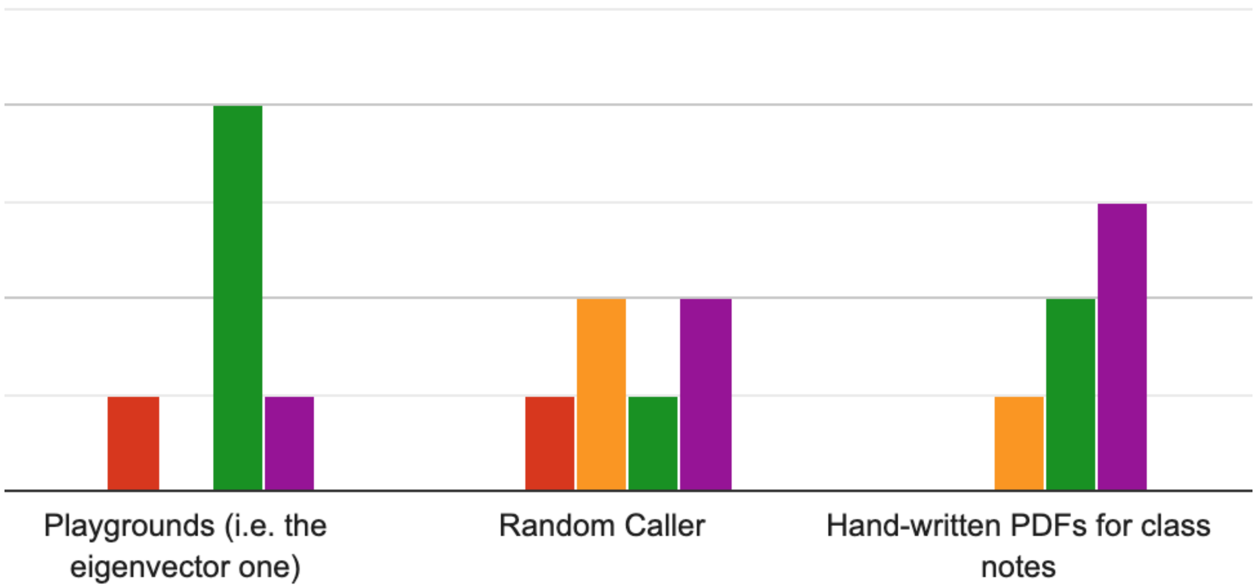

Figure 1b: blue = 1, red = 2, orange = 3, green = 4, purple = 5. Hated is 1. Loved is 5.

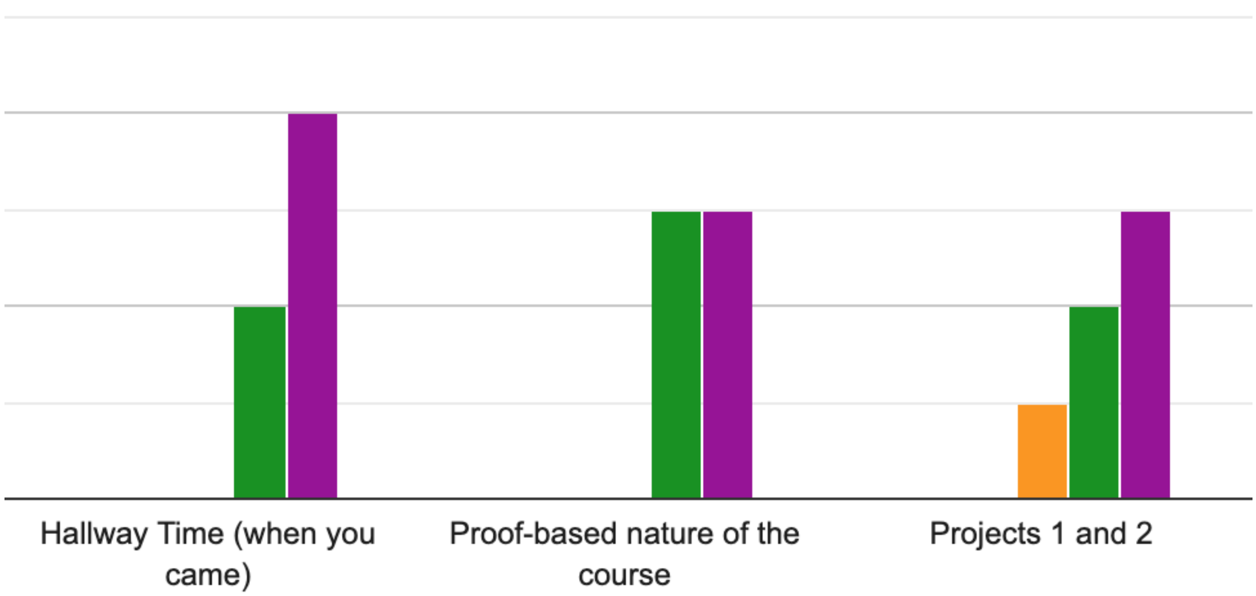

Figure 1c: blue = 1, red = 2, orange = 3, green = 4, purple = 5. Hated is 1. Loved is 5.

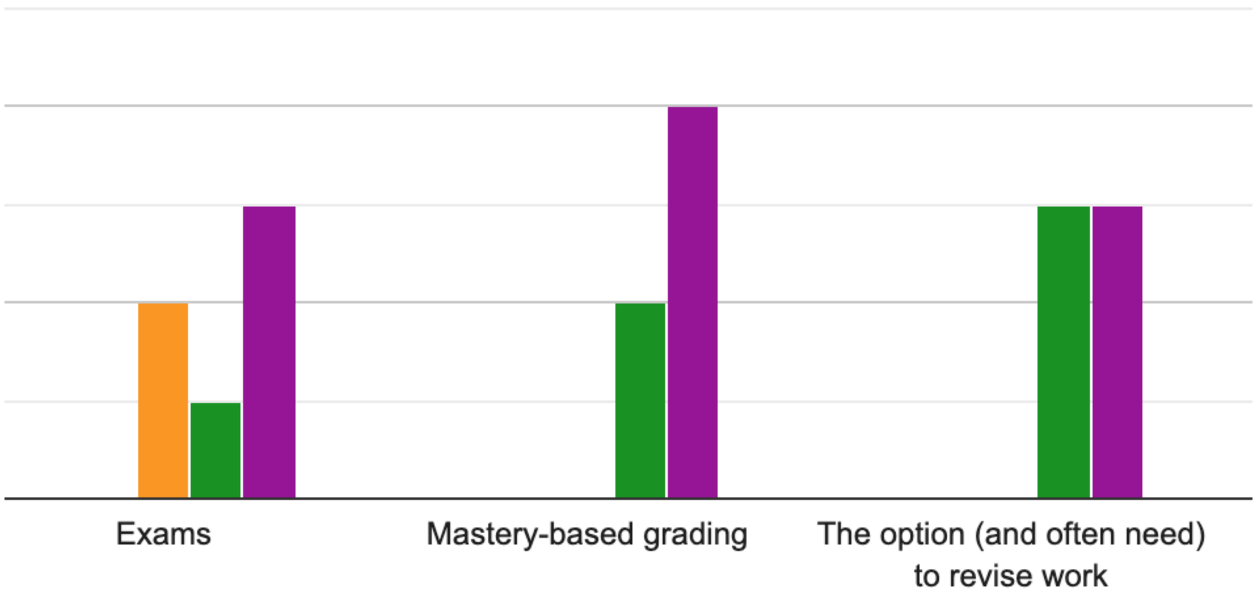

Figure 1d: blue = 1, red = 2, orange = 3, green = 4, purple = 5. Hated is 1. Loved is 5.

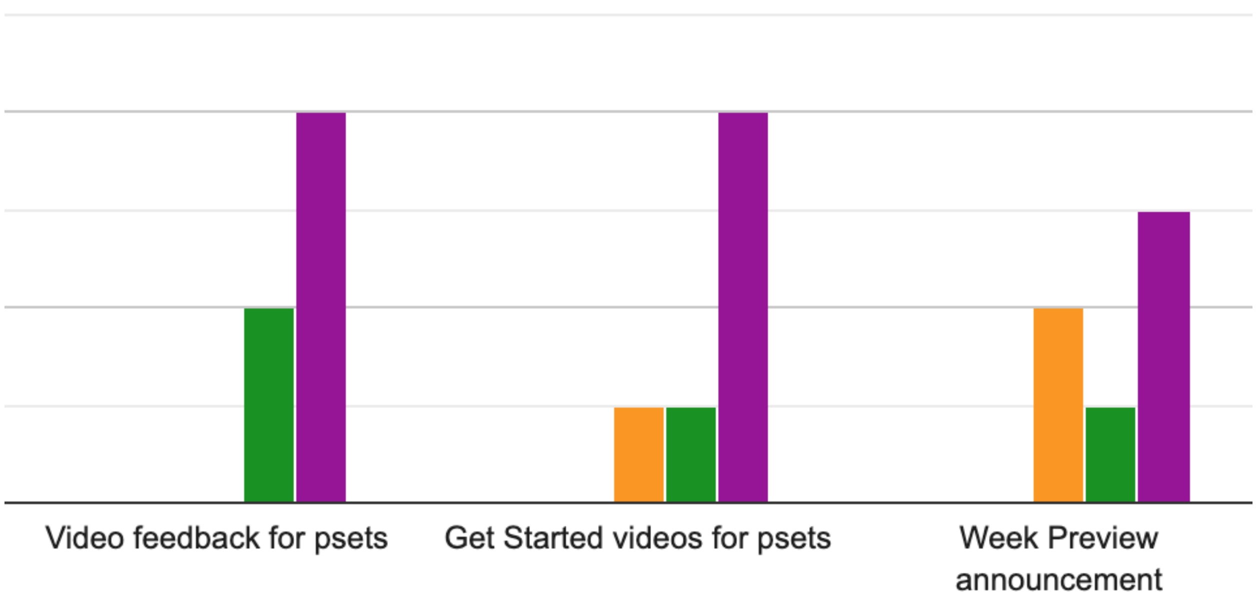

Figure 1e: blue = 1, red = 2, orange = 3, green = 4, purple = 5. Hated is 1. Loved is 5.

We also asked students to rate how much different factors contributed to them turning in work on time. Here, 1 = not important, 5 = critically important.

Figure 2: factors contributing to turning in work.

We asked students whether they agreed with the following statements:

(6/6) The Canvas page was organized effectively.

(3/6) I attended Hallway Time more because of the weekly joke.

(2/6) I attended Hallway Time more because of the rebrand away from Office Hours.

(3/6) The Week Preview announcements on Sunday helped me keep track of my work.

(1/6) The lessons on Monday were ineffective because they repeated material we had to read anyway in the textbook.

(2/6) I used the recordings of our class times.

(5/6) The acronyms of the exams (I.M.P.S., S.M.I.L.E., S.E.E.Y.A.L.A.T.E.R.) made them feel friendlier.

(5/6) I enjoyed that I could explore my own interests for the projects.

(4/6) I got more excited about linear algebra after listening to other student projects.

(0/6) There was too much work in this class.

(6/6) I have a good mental model of what a linear transformation looks like.

(6/6) I learned a lot this semester and grew as a mathematician.

(5/6) I would have preferred if the class were offered in person (assuming COVID didn't exist).

We asked students to rate different aspects of the instructor: accessible, effective lectures, enthusiastic, useful feedback, returns assignments. All students gave the maximum score of 5 for all categories, except one student rated “effective lectures” as a 4.

We asked students how many hours they spent weekly on coursework outside of class.

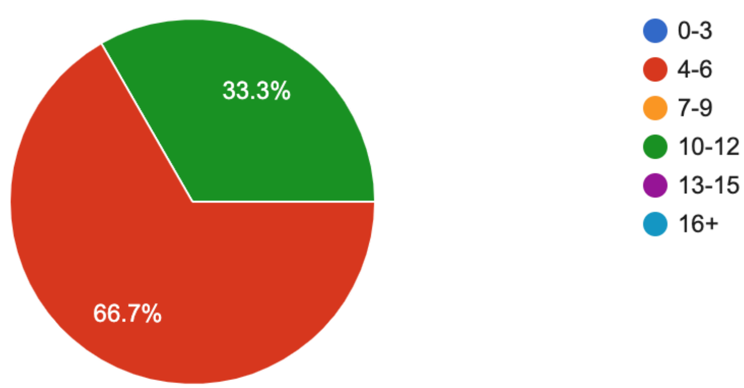

Figure 3: two students spent 10-12 hours per week outside of class; the rest spent 4-6.

The final question asked students to optionally write a testimonial for inclusion in this paper about their experience in the class. Five out of six students provided one. Comments are reproduced exactly as we received them.

> I loved Laker's Linear Algebra course because of the ways that it expanded my insight into the power of math. Over the duration of the class, I grew to understand concepts such as nilpotent operators and eigenbases that I never thought I would have understood. I've proved rules and formulas that I never thought I could. I've explored the possibilities of Linear Algebra to topics of my interest, such as computer vision and artificial intelligence. Overall, I would rate this class a 10/10 experience. Thank you Laker!
>
> I really enjoyed learning about Linear Algebra, the focus on proofs made things ten times more interesting and Laker's explanations were really helpful when the textbook was hard to follow, which itself was rare! This was a great class to get my feet wet with higher level abstract math, and I would enthusiastically recommend it to anyone who is curious about higher level math, especially if repetitive calculations isn't what piquing your interesting (sic).
>
> I thought Laker's teaching style was a breathe (sic) of fresh air compared to teachers!
>
> The "get started" videos were immensely helpful! They were just generic enough to encourage me to solve the problems independently, but simultaneously just specific enough to give me a stepping stone to guide me to the right path. Case-in-point, I frequently used your videos for a visual intuition behind the problem and what the question was asking in-particular -- then I could solve it using the foundations I had gathered.

> The course was great! The problem sets were good measures of what was in the book, the lectures were interesting and easy to follow along, the video feedback was incredibly helpful and I hope our other teachers adopt that as well.

We also asked the co-teacher, Chris, to summarize his experience with the class.

> Being able to help mentor Laker through his creation and instruction of his Linear Algebra class has been a complete joy. His relaxed and friendly approach, mixed with his firm grasp of the material, helped keep the students motivated and enjoying the class even when the material was challenging. Laker set out clearly defined goals for success in the class, so everyone was clear on what they needed to do to succeed, and he was always there for one-on-one help whenever they needed it. He used various tools and resources to keep the class interested and allow his students to gain an intuitive and deep understanding of the material. His use of Loom for grading and KA's playground for visualizing material were innovative and effective approaches to grading and building a more profound mastery of core concepts. His classroom instruction was clear and well-articulated while leaving space for the students to absorb the material and get their questions answered. He kept the class engaged by using a mixture of resources such as videos, textbooks, guest lectures, well-prepared examples, and fun projects that allowed them to explore how linear algebra applies to topics of their interests. I learned a lot by working with Laker and will use many of his techniques in my teaching in the future.

Again, all materials created for the course are available in Appendix A.

Chapter 4: Analysis and Discussion

The survey indicates which aspects of the course the students found valuable. For the first question, we can categorize student responses by the average rating as hated, disliked, controversial, liked, or loved. We do so by rounding the average to a number from 1 - 5 corresponding to the categories. We also include the binary responses from the third question, interspersed with quotes from the student testimonials.

Students loved using Sheldon Axler's textbook *Linear Algebra Done Right*, loved the theory-based approach, and loved the proof-based nature of the course. This is heartening evidence that de-emphasizing rote matrix calculations improves how much students enjoy learning linear algebra. It is also testament to the ability of students who have no background in proofs to quickly pick up the skill while learning linear algebra simultaneously.

Students also loved the animated videos from the YouTube channel 3blue1brown with which we began the course. Although one student rated these neutrally, the rest rated it as liked or loved. This is evidence that the videos helped the students to internalize linear algebra visually. Perhaps these videos are the reason that 6 out of 6 students agreed they had a good mental model of what a linear transformation looks like at the end.

Students liked the interactive playgrounds. While one student found them unhelpful, the vast majority of students rated it as helpful. However, only one student rated it as highly as possible. Perhaps this is evidence that we should incorporate the playgrounds into more of the course. As is, they played a minor role outside of a few key lessons.

Students liked the random caller, but this was the most controversial rating as measured by the high variance in responses. Perhaps this is no surprise: the random caller forces students to remain alert at all times in class, and it may cause stress to be cold called. Although we tried to reduce this social anxiety with a friendly atmosphere, the students who disliked the random caller are justified in feeling this way. What is certain is that the random caller helped us while instructing to keep in touch with what students understood at a given time; if a randomly called student expressed bafflement at a simple question, it was a cue to slow down the pace of the lecture.

Students liked the hand-written notes we used for lectures. This suggests that our attempts to include examples and, in particular, questions to ask students to keep them engaged succeeded in making lecture more valuable. However, it also suggests room for improvement in how we maximize student interactivity in class. One limitation was that opening breakout rooms in Zoom while recording was impossible. Future teachings of this class that are in-person, do not plan to record, or can record more robustly could increase interactivity through more small-group discussions, relying less on a lecture format with the hand-written notes. Another possible reason that students would not like hand-written notes was if they felt the notes simply repeated material they had to read in the textbook anyway. When asked, however, only 1 out of 6 students agreed with this statement. This is compelling evidence that students preferred the twice-over treatment of material, with enough variation, examples, and interactivity to not feel like we were reading the textbook verbatim.

Students loved Hallway Time. Half of students agreed they came to Hallway Time more often because of the weekly joke. Only a third of students thought the rebrand away from office hours contributed to them coming. This is evidence that some of the effort we put into enticing students to come to Hallway Time may have been unnecessary. An alternative explanation, however, is that some subset of students will not attend office hours consistently no matter what, so we should do what we can to at least entice the rest. Under this explanation, the time to draft weekly jokes is worth it. It is even worth it to rebrand away from office hours, if only because this takes no effort. For future teachers who can use the jokes we wrote, this humor component will also take much less time. Finally, notwithstanding the effect of the jokes and friendly rebrand on office hour attendance, they serve the inherently valuable goal of creating an inclusive environment for all students.

Students liked Projects 1 and 2. This is largely due to the freedom afforded to students to explore whatever interested them, which 5 out of 6 students agreed they enjoyed. Another benefit was that students could hear diverse applications of linear algebra during presentations, which 4 out of 6 students said made them more excited to learn linear algebra. We will note, however, that the mathematical presentation skills of the students were sometimes poor. With freedom came lack of oversight, which removed an opportunity for us to work with students to craft effective presentations. It is because communication skills are so central in math that we changed the model for Project 3, opting for a seminar-style discussion expressly aimed at building mathematical communication skills.

Students liked the exams. It is no wonder these did not receive the highest ratings because nobody loves tests, so we interpret it as a success that students rated exams this highly. Five out of six students agreed that the acronyms made the exams feel friendlier. Our goal with the exams was to test basic proficiency while affording more flexibility in how students could earn mastery in a given unit.

Students loved the Get Started videos. A common issue for students is overcoming the initial hurdle of understanding what a problem is even asking; this is evidence that the Get Started videos helped alleviate this obstacle. The potential pitfall with these videos is revealing solutions to the problems. However, this did not bear out to be an issue. As one student wrote, “The ‘get started’ videos were immensely helpful! They were just generic enough to encourage me to solve the problems independently, but simultaneously just specific enough to give me a stepping stone to guide me to the right path.”

Students also loved mastery-based grading, video feedback for their problem sets, and even the option (and often need) to revise work. This is exciting evidence that students found one of the core tenets of our pedagogy highly effective. One student said, “The video feedback was incredibly helpful and I hope our other teachers adopt that as well.” Video feedback has the added benefit of being pausable and rewindable. Equipped with video feedback, students can almost always revise a challenging problem set to proficiency with no further oversight, saving time for the teacher. Despite the potential for revisions to feel like unwanted extra work, a resounding 0 out of 6 students said that there was too much work in the class. One third of students spent 10 - 12 hours per week on the class and the remainder spent 4 - 6 hours per week.

Students liked the Week Preview announcement, which contained reminders about upcoming deadlines. Only half of students, however, agreed that these announcements helped them keep track of their work. The effort to create Week Preview announcements is low, so they remain a valuable tool to make deadlines as clear as possible to students.

All six students agreed that the Canvas page was effectively organized, which is statistically significant assuming a binomial distribution of opinions (p-value of 0.015). We made an effort to remove clutter, which reflected our general practice to focus on a smaller number of more important assignments.

Only a third of students used the recordings of classes. This makes recording lectures a lower priority way to spend time, especially given the abundance of high-quality lessons on linear algebra topics already freely available on the internet. We produced recordings despite their low usage rate for inclusion in this paper.

Students also resoundingly thought the instructor was accessible, was enthusiastic, had effective lectures, had useful feedback, and returned assignments. All students rated all these categories 5 out of 5 except for one who rated “effective lectures” as a 4. We made an effort to return assignments promptly and to be available to students for 1:1 assistance. It is gratifying that these efforts were reflected by student responses. As the co-teacher Chris wrote, “Laker set out clearly defined goals for success in the class, so everyone was clear on what they needed to do to succeed, and he was always there for one-on-one help whenever they needed it."

Our analysis based on the survey has several limitations. First, the students have learned linear algebra no other way than ours: because we have no control group, we cannot reliably compare our pedagogy to other approaches. All we can say is whether our ideas yielded student satisfaction or not. Furthermore, a sample size of 6 is prone to noise. We should make cautious conclusions based on the data. The students in the class were also all friends with each other and with the instructor. Additionally, the students self-selected by a prior interest in taking linear algebra. If the class were taught again among total strangers, it would remain to see whether strategies such as rebranding office hours would be more or less valuable. Because the class was taught at Khan Lab School, students may have also been biased toward enjoying the mastery-based approach taken to grading.

Notwithstanding, we conclude from the survey that the most valuable components of the course were to emphasize theory over computation with Axler's textbook, to begin with animated videos from 3blue1brown, to hold fun office hours, to create Get Started videos for challenging problem sets, and to deliver video feedback for revisions within a framework of mastery-based learning. Less valuable components which might need to be rethought were the playgrounds, the random caller, the projects, the exams, and the Week Preview announcements. Even these all scored a rating of "liked" at the worst: affirmation that the course overall found great success.

Future work could replicate the course we taught at a larger scale to compare the effectiveness of these strategies when the teacher's time per student is more limited. Future work could also compile resources for other math courses, as we did for linear algebra. There is surely a wealth of literature for other courses that we have not touched on. Future research could also test the strategies we implemented here when used to teach other math courses. How to best teach math is an open question ripe for further investigation.

Chapter 5: Reflection

The start-to-finish process of proposing the linear algebra course, securing approval, brainstorming how it would run, then teaching it and creating policies to keep students on track drew from many diverse skills I have learned throughout high school. Most importantly, I had the help of many friends and family members who provided consistent support to me.

The obvious skill I drew upon was knowledge of the content of linear algebra. I first learned linear algebra as an 8th grader with 3blue1brown's series, after which I took an official course in 10th grade. That course was MIT 18.700, the original model for this course. The time between finishing that course and beginning to teach this course allowed me to more deeply grasp the material, which is why I was qualified to teach the subject in my senior spring. Additionally, I spent the summer of my sophomore year developing practice problems for Khan Academy's multivariable calculus course. This experience taught me about course design, let me explore the benefits of teaching math from a visual perspective, and surrounded me with colleagues who were passionate about math education.

But I could not have taught anything if not for Chris Lippi's enthusiasm and our Head of Upper School Kim Dow's approval. To earn these, I had to deliver a pitch of my idea. My mother provided invaluable assistance crafting these pitches by considering what Kim's and Chris's potential concerns would be. Working closely with Kim and our college counselor, Janan Sabeh, we created clear expectations for students in the form of a contract to ensure none fell behind in this experimental course.

In addition to Chris, my mentor from Khan Academy, Jeff Dodds, became a primary course advisor. Having taught AP Statistics for over a decade, he offered guidance for how to lead effective lessons and engage students. These are skills for which any new teacher is grateful. My friend Drew Bent also acted as a course advisor, helping in particular to brainstorm what kind of exam would be most helpful for the students, or whether exams were a good idea at all. The acronymed exams are the result.

When multiple students did not turn in the first problem set by the deadline, I faced the question of how to deal with this. The matter was delicate because a mastery-based system is oddly all-or-nothing, like pass/fail/revise. For seniors who needed to graduate on time, revisions into the indefinite future were not an option. Therefore, with the support of my parents, I gained Chris's approval to institute a policy that no late work would be accepted; students needed to at least turn in partial solutions on the night of the due date. The policy worked wonders and wouldn't have been possible without the creativity and insight of those around me.

The takeaway from these acknowledgments is how critical personal connections were to this Capstone Project. Even the students chose to buy into the idea and work hard. I could not have offered this course alone and am indebted to everyone who helped to make it a reality.

At the same time, I am pleased with my role in the endeavor. I am glad the students found me as effective an instructor as they indicated on the survey. I tried hard to return assignments quickly and to implement in practical ways the ideals I have for math education, like intuitive visuals and theory over calculation. I also made a point of asking for feedback consistently throughout the course and adjusting based on this feedback. I spent time in the ways that directly reached students, like creating video feedback and working 1:1 with struggling students when they needed it.

My takeaway here is that the recipe for success is sustained effort in the right direction. Sustained effort is a matter of perspiration, but finding the right direction is rarely a bolt of inspiration. More often, I found the right direction by being surrounded by the right people. Ideas trickle in slowly over time, so the best plan is to work with what one has at the time, while being open to replacing an ineffective strategy with a more effective one.

I am beyond thankful for all the support I received working on this project. To maximize its future impact, I now recap the conclusions we can draw and lay down recommendations for how to implement these conclusions in other math classes.

Our class focused on theory and deemphasized calculations, introduced fundamental concepts early on with animations, and offered computer programs to play with the material. Students explored applications interesting to them. Mastery-based learning and video feedback formed the backdrop.

These innovations succeeded. We should apply them beyond linear algebra.

Starting as early as basic algebra and through calculus, teachers who deemphasize calculations in favor of theory and concepts may find that their students understand material more deeply. While calculations are indispensable, they should not entirely comprise a course, as is too common in American math education.

Introducing fundamental concepts from throughout the course early gives students a roadmap of what they will learn. Too often, a student lives in the present in a math course. They know what will be covered today and tomorrow but do not know about in one month or two. As a result, students must develop rushed mental models for everything they learn on the spot. Often, students develop no intuition at all, subsisting on rote memorization. Introducing fundamental concepts early allows students time to begin pondering the objects and deeper questions of a subject at the very beginning. In linear algebra, this means the teacher can introduce eigenvalues as an old friend. This improves student understanding of material; no matter how poor their rigorous mental model at first, a student will at least intuit the broad strokes of a concept by recalling how it was introduced at the beginning.

Paper is a dated medium for conveying mathematics. New technologies allow students to pull vectors around, slide variables, and see instant visual feedback all the while. We only rarely used these kinds of playgrounds in our course, but on the scarce occasions when we did, they afforded astonishing understanding. Compared to a half hour of lecture, a half hour of active engagement with the playground yielded better student comprehension. This was the case in Lecture 14 on Eigenvalues and Eigenvectors. Teachers should seek out ways for their students to engage with math interactively.

Students ought also to be given freedom to explore what interests them. The applications of linear algebra are nearly limitless, so there is no shortage to choose from. Students must only be guided as to what to look for. The primary benefit of this freedom is when students present their work to the class. Like in a mini math conference, the students see diverse applications of the subject beyond what they personally have explored. While the applications are more limited and may be less interesting in a basic algebra course, they still exist. The most tantalizing applications here perhaps exist in physics.

Finally, the pedagogical backdrop of how grading functions is perhaps the most critical motivation to most students. A mastery-based approach removes stress from students because all work can be revised. Video feedback is one of the most efficient ways to increase the rate at which teacher time spent grading translates to student time spent learning. In fact, it makes this ratio nearly one to one! Standard A-F grading for each assignment leaves the possibility for students to scrape by early on, hindering their foundational understanding. One must eventually move on, but when possible, revising work to earn mastery greatly deepens a student's understanding of the material. In linear algebra and other proof-based courses, revisions also offer a chance for the teacher to instill proper mathematical writing skills.

I would urge Khan Lab School and other institutions to implement any and all of these ideas in their own math courses. For linear algebra in particular, our duty as mathematicians is to teach a class fit for the 21st century so that our students can go forth understanding linear algebra not as a frustrating black box but as a rich toolkit that will help them to solve the world's greatest problems.

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Appendix A

All material we created for the class is reproduced or linked below: syllabus, lecture notes, weekly surveys, problem sets, exams. Recordings of the lectures are available here. By lecture:

Lecture 1 | Why Linear Algebra and Course Overview
Lecture 2 | Discussion of Essence of Linear Algebra Chapters 1-6
Lecture 3 | Intro to Set Theory, Quantifiers, and Proofs
Lecture 4 | Discussion of Essence of Linear Algebra Chapters 7-9 and 13-15
Lecture 5 | Vector Spaces and Intro to Axler
Lecture 6 | Subspaces, Direct Sums, Span, and Linear Independence
Lecture 7 | Markov Chains Guest Lecture
Lecture 8 | Basis and Dimension
Lecture 9 | Linear Maps, Null Space, and Range
Lecture 10 | Student Presentations
Lecture 11 | Matrices and Inverses
Lecture 12 | Gaussian Elimination
Lecture 13 | Polynomials
Lecture 14 | Eigenvalues and Eigenvectors
Lecture 15 | Quotient Spaces
Lecture 16 | Eigenspaces and Diagonalization
Lecture 17 | Inner Product Spaces
Lecture 18 | Gram-Schmidt and Minimization
Lecture 19 | Adjoints and the Spectral Theorem
Lecture 20 | Student Presentations (1/2)
Lecture 21 | Student Presentations (2/2)
Lecture 22 | Positive Operators, Isometries, and the Polar Decomposition
Lecture 23 | Generalized Eigenvectors
Lecture 24 | Characteristic Polynomials and Jordan Form
Lecture 25 | Recommender Systems Guest Lecture
Lecture 26 | Singular Value Decomposition Seminar
Lecture 27 | Determinants
Lecture 28 | Higher Math Beyond Linear Algebra I
Lecture 29 | Higher Math Beyond Linear Algebra II

KLS Linear Algebra: Syllabus

Description: This course offers a rigorous treatment of linear algebra, including vector spaces, systems of linear equations, bases, linear independence, matrices, determinants, eigenvalues, inner products, and the spectral theorem. Emphasis is placed on theory and proofs. The prerequisite is calculus.

Mastery: Students must reach proficient in all units to reach mastery. To reach proficient in a unit, students must reach proficient in at least two modalities. If any modality is in progress, the unit is in progress. If any unit is in progress, the student cannot reach mastery.

Unit	**Modality 1: Problem Sets**	**Modality 2: Exam**	**Modality 3: Project**	**Mastery Level**
1: Vector Spaces and Linear Maps				
2: Matrices and Eigenvectors				
3: Inner Product Spaces and the Spectral Theorem				
Total				

Example projects:
Unit 1: Research an interesting vector space. Present to the class. Example: positions in the game Lights Out form a vector space.
Unit 2: Make a Markov chain, diagonalize it, predict long-term behavior, write 300 words explaining the work. Present to the class.
Unit 3: Pick a fun function, approximate it via least squares. Present to the class. Bonus: write a program to automate the process.
Students can always propose their own project.

Resources:
Textbook: *Linear Algebra Done Right* by Sheldon Axler. Get the third edition. The author made videos to supplement the textbook.
Visual Intuition: *Essence of Linear Algebra* by 3blue1brown. Watch at 1x speed the first time.
Extra Reading: supplementary notes from MIT OCW 18.700. [1a] Gaussian elimination.

Schedule: There are nineteen academic weeks. Each week has two one-hour class times (Mon 9:00, Wed 9:00). Monday is dedicated to mini-lectures, student questions, and examples of how to solve exercises in Axler. Wednesday is supported problem solving time or exam time, except that for the first two weeks it is like Monday. Homework is due Sunday at 11:59 the following week.

Week #	**Content**	**Homework**
Week 1	syllabus and course outline	Order textbook. Watch *Essence of Linear Algebra* chapters 1-6.
Week 2	sets, notation, and proofs	Watch *Essence of Linear Algebra* chapters 7-15.
M1 Week 3	vector spaces, properties	Read Axler sections 1A, 1B.
Week 4	subspaces, span, linear independence	Read Axler sections 1C, 2A. Problem Set 1 due.
Week 5	bases, dimension	Read Axler sections 2B, 2C. *Project 1 due*.
Week 6	linear maps, null space, range	Read Axler sections 3A, 3B. Problem Set 2 due.
M2 Week 7	matrices and invertibility	Read Axler sections 3C, 3D. *Exam 1.*
Week 8	finite fields, Gaussian elimination	Read notes [1a]. Problem Set 3 due.
Week 9	eigenvalues and eigenvectors	Read Axler section 5A, 5B.
Week 10	eigenspaces and diagonalization	Read Axler section 5C. Problem Set 4 due.
M3 Week 11	inner products and norms	Read Axler section 6A. Problem Set 5 due. *Exam 2*.
Week 12	orthonormal bases and minimization	Read Axler sections 6B, 6C. *Project 2 due.*
Week 13	adjoints and the spectral theorem	Read Axler section 7A, 7B. Problem Set 6 due.
Week 14	positive operators and the SVD	Read Axler section 7C, 7D.
M4 Week 15	generalized eigenvectors	Read Axler section 8A, 8B. Problem Set 7 due.
Week 16	minimal polynomials and Jordan form	Read Axler section 8C, 8D. *Exam 3*.
Week 17	trace and determinant	Read Axler sections 10A, 10B. *Project 3 due.*
Week 18	buffer, review, or bonus material	
Week 19	buffer, review, or bonus material	

Lecture Notes

Week 1: slides only
Week 2: Set Theory Intro
Week 3: slides only
Week 4: Subspaces, Direct Sums, Span, and Linear Independence
Week 5: Basis and Dimension
Week 6: Linear Maps, Null Space, and Range
Week 7: Matrices and Inverses
Week 8: Gaussian Elimination, Polynomials
Week 9: Quotient Spaces[22]
Week 10: Eigenspaces and Diagonalization
Week 11: Inner Product Spaces
Week 12: Gram-Schmidt and Minimization
Week 13: Adjoints and the Spectral Theorem
Week 14: Positive Operators, Isometries, and the Polar Decomposition
Week 15: Generalized Eigenvectors
Week 16: Characteristic Polynomials and Jordan Form
Week 17: Determinants
Week 18: slides only

All slides for the course are collected here.

Weekly Surveys

3b1b Survey [Episodes 1-6]
3b1b Survey [Episodes 7-9 and 13-15]
Axler Survey [Sections 1A and 1B]
Axler Survey [Sections 1C and 2A]
Axler Survey [Sections 2B and 2C]
Axler Survey [Sections 3A and 3B]
Axler Survey [Sections 3C and 3D]
Packet Survey [Gaussian Elimination]
Axler Survey [Sections 5A and 5B]
Axler Survey [Section 5C]
Axler Survey [Section 6A]
Axler Survey [Sections 6B and 6C]

[22] We do not have handwritten lecture notes for an introduction to eigenvectors, because this lesson centered around discussion, slides, and using the eigenvector playground.

Axler Survey [Sections 7A and 7B]
Axler Survey [Sections 7C and 7D]
Axler Survey [Sections 8A and 8B]
Axler Survey [Sections 8C and 8D]
Axler Survey [Sections 10A and 10B]

Problem Sets

Problem Set 1
Problem Set 2
Problem Set 3
Problem Set 4
Problem Set 5
Problem Set 6
Problem Set 7
Problem Set 8 (optional)
Problem Set 9 (optional)

Exams

Exam 1
Practice Exam 1
Solutions to Practice Exam 1

Exam 2
Practice Exam 2
Solutions to Practice Exam 2

Exam 3
Practice Exam 3
Solutions to Practice Exam 3

Appendix B

We reproduce the code of the random caller below. Names have been anonymized. The purpose of the variable pool_size is to not call on a student who has recently answered. Whenever the program calls on a student, the name enters a queue which prevents it from being called on again until enough other names have been called. The value of pool_size is the number of students the program may call on at any given time.

```
import random

names = ['Student 1', 'Student 2', Student 3', 'Student 4', 'Student 5', 'Student 6’]

pool_size = 3 # size of active pool of possible answerers
last = random.sample(names, len(names) - pool_size)
current = last[0]

while True:
    input('')
    while current in last:
        current = random.choice(names)
    last.pop(0)
    last.append(current)
    print('> ' + current, end='')
```

Appendix C

We reproduce the jokes we told each week in class.

Week 1: no joke

Week 2: no joke

Week 3: *“Why did the man not want to get onto the bus crowded with passengers who weren't wearing masks? Because it was a dangerous vector space.”*

Week 4: *“What do you call an eigenvector that scales to make 100 million vaccines? A Bidenvector!”*

Week 5: *“Did you hear why Jeff Bezos is leaving Amazon? Some people say it’s because he wasn’t good enough at linear algebra. But at the press conference, Bezos said that was a basis-less claim.”*

Week 6: *“Why did mathematicians at Moderna use linear algebra to help develop a vaccine? Because they needed to make it injective.”*

Week 7: *“Why did the British outlaw studying linear algebra in 1776? Because they didn’t want any of their royal colonies to declare Linear Independence.”*

Week 8: *“Gauss’s crew team won first place two years straight, but the third year he was injured and off the team. Why did the crew team stop winning? Because after Gaussian elimination, they only had tiny, reduced rows!”*

Week 9: *“Why did the eigenvector girlfriend break up with her eigenvector boyfriend? Because when she discovered their eigenvalues were fundamentally different, she knew she needed her linear independence.”*

Week 10: *“How did Harry Potter know where to go on the map of London to buy his school supplies? He Diagonalleyized the map!”*

Week 11: The joke is too long to fit here; see the end of the appendix.

Week 12: *"Why was the patient angry at the orthodontist? When she took off her braces, her teeth were all at 90° angles."*

Week 13: *"What did the Salem Witch Trials have in common with linear algebra? The Puritans were just applying the Spectral Theorem: if she doesn't look normal, then burn her alive!"*

Week 14: *"When I was young I wanted to become pregnant, and actually I recently took a pregnancy test. To my delight, I saw a little red plus sign! I was so excited! But unfortunately, my doctor told me that, because I was a boy, the result was only positive semidefinite."*

Week 15: *"You all know what to call a linear operator which equals zero when raised to a high enough power: nilpotent. But what do you call a fish that can breathe well? Gilpotent. What about a costly repair or a drug that'll put you right to sleep? Billpotent and pillpotent! An effective refrigerator or a loud, sharp scream? Chillpotent and shrillpotent. An herb to seriously spice up your eggs? Dillpotent. And, finally, a mountain? Hillpotent."*

Week 16: *"It's a little known fact that Michael Jordan owns a parrot. Why is this parrot so skinny? Because its name is Poly No Meal, and it's a minimal Poly No Meal at that! This parrot is also the best at dunking, no doubt because it's in Jordan Form."*

Week 17: *"Why did the detective abandon the investigation into the crime of the diagonal matrix? Because when he added everything up, he found only trace amounts of evidence."*

Week 18: *"Hey guys, I've really enjoyed teaching linear algebra, so I've decided to adopt a new nickname. Want to know what it is? You can call me LA Laker... for linear algebra Laker!"*

The joke for week 11 was quite long, so we reproduce it away from the rest:

> "Hey [name], did you know your refrigerator is running? Well you better go and catch it!
>
> Okay, but getting serious here. How do you put a linear algebra professor into the refrigerator? Step one: open the door. Step two: put him in. Step three: close the door.

Okay, now say it's an hour later. How would you put an inner product in the fridge? Open the door. Take the professor out. Put the inner product in. Close the door.

Now, how would you put in a magnet? No, no, no! You don't put magnets in the fridge! You stick them on the door. And anyway, the inner product is inside so we don't currently have any notion of magnet-tude.

All right, now let's say there's a fisherman far away in the Atlantic ocean. Why can't he catch any fish? Because the inner product is still in the refrigerator! And how can you expect an angler to catch fish if he has no sense of angle?

Okay, finally some students are waiting to join their linear algebra class, but the teacher is late: one minute, three minutes, five minutes. Why doesn't the teacher show up? Because you all put him in the fridge, remember?! He's practically hypothermic. Way, way too cool for school."

Some jokes went unused or were told outside of Hallway Time:

"Why is Google using its headquarters as a vaccination site? It wanted to have the distributive property."

"What happens when a linear algebra professor switches jobs to work in a meat packing factory? He gathers up the scraps and makes millions selling them as canned 'Span.'"

"What did the teenage eigenvector tell her overprotective parents? Leave me alone, I need my eigenspace!"

"What do you call the nest of an ant colony that has dug really deep into the ground? An in-very-ant subspace!"

"The little eigenvector wanted to be an astronaut when he grew up, but he couldn't pronounce such a long word. What did he say instead? When I gwow up, I geh in spaaace! (eigenspace)"

"If you're Sheldon Axler, what do you say at the steakhouse? I'd like my steak... done right."

The lightning introductions to higher math from the final classes had a joke each:

Abstract algebra: In the abstract algebra class, what do they call the student who scores highest on the second midterm? Lord of the Rings.

Field theory: Galois died in a duel at the age of 20. It wasn't his field of expertise.

Markov chains: A drunk man will always find his way home, but a drunk bird may be out of luck.

Topology: What is a topologist's favorite breakfast? Froot Loops with a donut and coffee mug, although she can't tell the difference.

Multivariable calculus: What did the multivariable calculus professor say to his students before the final exam? No pressure, but the Stokes are high.

Measure theory: The measure theorist paid the baker in pennies, nickels, dimes, quarters, dollars, fives, then tens until he was done, rather than the price of each item one after the other. What did the measure theorist purchase? Lebesguels!

Graph theory: What did the graph theory professor say to his students after they all aced their final exam? Congraphulations!

Theory of computation: What did the programmer say when faced with solving a size n travelling salesman problem for a grid of scenarios of length m and width a? O(man!)

Finally, the very last class included a rap about the four subjects we covered that day.

Uh, uh. Here's a math rap.
Uh, uh. When I'm done please clap.

Two variables in a function I got
Two numbers in the output slot.
How do I take a derivative
What numbers must I give?
Turns out the answer's slick
Just use a square matrix.
That's right, a linear map
Straightaway fills in the gap.
This local approximation
Is an epsilon-delta generalization.
Nudge a function to see it move
And use open sets this to prove.
From there is the question of integration
Where Stokes gives us cause for elation,
Since functions on boundaries evaluated
Equal derivatives everywhere else calculated.

Uh, uh. Movin' on.

See, I got a problem with integration.
What if my dart can't decide its location?
Is it rational, or is it not?
This is getting steamy, the debate is hot.
To the rescue is the man Lebesgue.
His measure is everyone's fav.
It says the length of a set

Is the infimum of its covering boxes.
With it, our integral we can get
No worries about paradoxes obnoxious!

Uh, uh. Movin' on.

You might wonder what nodes and edges can do.
Just about anything: it's true!
We got paths and cycles and neighbors and degrees,
Minors, marriage, coloring, and spanning trees!
Turns out any preferences are okay
You can always get matched with no foul play.
So it's a whole wide world to explore,
And any map needs colors just four.

Uh, uh. Movin' on. Uh, uh. Last one!

So you're name is Alan Turing
And your mind is brewing.
You have a model for computing
With a tape, head, and moving,
Its power is fabulous,
Same as lambda calculus.
But though it's miraculous,
Its reach isn't limitless.
See, the Halting Problem proclaimed
Computers are strictly maimed
Of the wondrous ability
To serve just any utility.
Thus to solve a given task,
We must ask
How long it will take
Or whether an algorithm is even possible to make.
And with big O notation
We can quantify duration.
Is it fast to solve, or merely fast to check?
If one then joy, if the other I'm a wreck.
But if you prove P = NP
You will earn a lot of money.

Uh, uh. With that the rap ends.
Uh, uh. Go learn math, friends.